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DETERMINATION OF THE BREAKING LOAD AND THE POSITION AND DIRECTION OF A FRACTURE USING THE GRADIENT APPROACH

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The results of an investigation, the purpose of which was to obtain answers to the question of where, in what direction, and for what load a fracture begins when there is a concentration of stresses, are presented. The gradient condition of stability is used to solve this problem. The application of this condition to the problem of the stretching of a plate with an elliptical opening, the major axis of which is inclined to the stretching axis, is considered. The results obtained are compared with experimental data in the literature on the fracture of plane specimens with inclined cracks. Good agreement is found between the theoretical and experimental data, and the universality of the two-parameter gradient conditions of stability considered is noted. The gradient condition can not only be used for crack-type concentrators but also in the more general case.

1. Discussion of the Problem. It would appear that, to determine the position where a fracture begins, it is sufficient to obtain the point in the body where the equivalent stress, assumed in one or another theory of breaking strength, reaches a maximum. However, the problem is not so simple. The point is that in the case of a nonuniform state of stress, in order to judge the breaking strength, it is necessary to know not only the value of the equivalent stress, but also the degree of nonuniformity of its distribution in the neighborhood of the point considered. In a number of papers [1-5], when investigating brittle fracture under static and cyclic loading, the first principal stress σ_i is used as the equivalent stress, while the measure of nonuniformity is taken to be the relative gradient of the first principal stress

$$g_1 = |\,\mathrm{grad}\,\sigma_1\,| / \sigma_1. \tag{1.1}$$

A nonuniform state of stress leads to a reduction in the destructive power of the stress at the point of maximum, or, in other words, to a reduction in its effectiveness. However, this can also be regarded as a situation in which the maximum stress at the instant when fracture begins exceeds the usual breaking strength σ_β, which is defined assuming a uniform state of stress [6-8]. For structural components with stress concentrators this phenomena finds reflection in the fact that the effective concentration factor is usually found to be less than the theoretical value. The gradient approach was used in [1] to estimate the effective concentration factor. In this paper we use the idea of the effective stress σ_e, to find which we propose to use the gradient approach

$$\sigma_e = \sigma_1 / f(g_1)$$

(σ_1 is the theoretical value of the first principal stress, $f(g_1) \geq 1$). When investigating brittle fracture the theoretical values of σ_1 are calculated from the elastic solution of the corresponding problem. The function $f(g_1)$ must essentially be exactly the same as the function which describes the increase in the local breaking strength in the gradient criteria proposed in [7, 8]. Consequently, using $f(g_1)$ of the combined two-parameter strength criterion, formulated in [8], we can write

$$\sigma_e = \sigma_1 / (1 - \beta + \sqrt{\beta^2 + L_1 g_1}). \tag{1.2}$$

Here L_1 is a parameter having the dimensions of length and which depends on the properties of the material, i.e. the characteristic dimension, and β is a variable parameter ($\beta \geq 0$), which can be regarded as an approximation parameter. Fracture begins at the point of the body considered when

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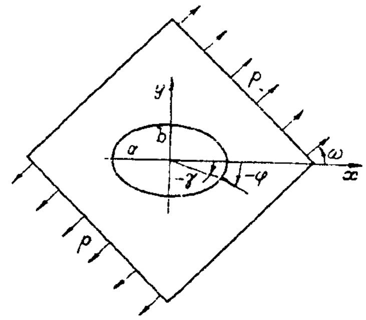

Fig. 1

$$\sigma_e = \sigma_w. \tag{1.3}$$

The parameter L_1 is found from the condition for the gradient approach to be compatible with linear fracture mechanics and is found from the following equation, obtained in [7]:

$$L_1 = \frac{2}{\pi} K_{1c}^2 / \sigma_w^2.$$

When this equation is satisfied the gradient breaking strength condition (1.2), (1.3) considered, in the special case of symmetrical crack-type stress concentrators, gives the same results as linear fracture mechanics.

However, for asymmetrical stress concentrators the question remains open. What the result obtained using the breaking-strength condition (1.2), (1.3) will be for asymmetrical problems of stress concentration and to what criteria of classical fracture mechanics they will most of all correspond, are unknown. Moreover, we do not know whether the maximum of the first principal stress σ_1 will be identical with the maximum of the effective stress σ_e. Note that for concentrators in the form of crack-like elliptical openings, and not mathematical cuts, the assumption that fracture begins at the point σ_1 does not agree with experimental data [9]. This was pointed out by MacClintock in the discussion in [9]. Hence, an answer to the last question also needs to be obtained since, according to the gradient breaking-strength condition (1.2), (1.3), fracture should begin at the point σ_e.

2. Analysis of the Asymmetrical Problem of Stress Concentration. The choice of problem. To answer the above questions we will consider the application of the gradient breaking-strength condition (1.2), (1.3) to the problem of the uniaxial stretching of a plate with an elliptic opening, the major axis of which is inclined at an angle ω to the stretching axis (Fig. 1). The elastic solution of this problem is well-known [10]. It was obtained in a special complex region ξ with polar coordinates ρ, θ and is given in terms of complex stress functions. We can change to x, y coordinates and is given in terms of complex stress functions. We can change to x, y coordinates (Fig. 1) by means of the conformal transformation

$$z = c(\xi + m/\xi). \tag{2.1}$$

Here $z = x + iy$; $\xi = \rho e^{i\theta}$; $c = (a + b)/2$; $m = (a - b)/(a + b)$. Using the first Kolosov formula in [10] we obtain the following expression for the sum of the stresses σ_θ and σ_ρ:

$$\sigma_\theta + \sigma_\rho = p \frac{\rho^4 - 2\rho^2\cos(2\theta - 2\omega) - m^2 + 2m\cos(2\omega)}{\rho^4 - 2m\rho^2\cos(2\theta) + m^2} \tag{2.2}$$

We will assume that fracture begins on the contour of the opening, where $\sigma_\rho = 0$ and $\tau_{\rho\theta} = 0$. Consequently, at those points of the contour where $\sigma_\theta > 0$, we have $\sigma_1 = \sigma_\theta$. It is these points that we are interested in when determining the point where fracture begins. Hence, using (2.2) and the fact that on the contour $\rho = 1$ we also have $\sigma_\rho = 0$, we can write

$$\sigma_1 = \sigma_\theta = p \frac{1 - 2\cos(2\theta - 2\omega) - m^2 + 2m\cos(2\omega)}{1 - 2m\cos(2\theta) + m^2}. \tag{2.3}$$

Determination of the Relative Gradient. The relative gradient g_1 can be found from (1.1). In order to use it we must first determine the modulus of the gradient of the first principal stress, which can conveniently be written in the form

$$| \operatorname{grad} \sigma_1 | = \sqrt{(\partial \sigma_1/\partial n)^2 + (\partial \sigma_1/\partial s)^2},$$

where $\partial\sigma_1/\partial n$ is the derivative along the normal to the contour of the opening, and $\partial\sigma_1/\partial s$ is the derivative along the tangent to the contour. On the part of the contour where $\sigma_1 = \sigma_\theta$, we have $\partial\sigma_1/\partial s = \partial\sigma_\theta/\partial s$. In order to obtain the derivative along the normal $\partial\sigma_1/\partial n$, we need to write an expression for σ_1 which holds over the whole region and then differentiate it. For a plane state of stress

$$\sigma_1 = \frac{1}{2}(\sigma_\theta + \sigma_\rho + \sqrt{(\sigma_\theta - \sigma_\rho)^2 + 4\tau_{\rho\theta}^2}).$$

Differentiation gives

$$\frac{\partial\sigma_1}{\partial n} = \frac{1}{2}\left(\frac{\partial\sigma_\theta}{\partial n} + \frac{\partial\sigma_\rho}{\partial n} + \frac{(\sigma_\theta - \sigma_\rho)\left(\dfrac{\partial\sigma_\theta}{\partial n} - \dfrac{\partial\sigma_\rho}{\partial n}\right) + 4\tau_{\rho\theta}\dfrac{\partial\tau_{\rho\theta}}{\partial n}}{\sqrt{(\sigma_\theta - \sigma_\rho)^2 + 4\tau_{\rho\theta}^2}} \right).$$

When $\sigma_\rho = 0$ and $\tau_{\rho\theta} = 0$ we have $\partial\sigma_1/\partial n = \partial\sigma_\theta/\partial n$. Hence, on the contour of the opening

$$| \operatorname{grad} \sigma_1 | = \sqrt{(\partial\sigma_\theta/\partial n)^2 + (\partial\sigma_\theta/\partial s)^2} = | \operatorname{grad} \sigma_\theta | . \tag{2.4}$$

We can write the derivative along the normal in the form

$$\partial\sigma_\theta/\partial n = \partial(\sigma_\theta + \sigma_\rho)/\partial n - \partial\sigma_\rho/\partial n. \tag{2.5}$$

Since $\sigma_\rho = 0$ and $\tau_{\rho\theta} = 0$ on the contour, it follows from the equations of equilibrium in cylindrical coordinates that

$$\partial\sigma_\rho/\partial n = \sigma_\theta/R$$

(R is the radius of curvature of the contour at the point considered). This equation was used in [11] but contained a misprint. The general equation for calculating the radius of curvature can be found, for example, in [12]. Using the coordinate θ as the parameter we can write

$$R = \frac{((\partial x/\partial\theta)^2 + (\partial y/\partial\theta)^2)^{3/2}}{| (\partial x/\partial\theta)(\partial^2 y/\partial\theta^2) - (\partial y/\partial\theta)(\partial^2 x/\partial\theta^2) |}.$$

In particular, for an elliptic opening

$$R = (a^2\sin^2(\theta) + b^2\cos^2(\theta))^{3/2}/(ab).$$

Reverting to Eq. (2.5) we can write

$$\frac{\partial(\sigma_\theta + \sigma_\rho)}{\partial n} = \frac{\partial(\sigma_\theta + \sigma_\rho)}{\partial\rho}\frac{\partial\rho}{\partial n}, \tag{2.6}$$

where the derivative with respect to the curvilinear coordinate ρ orthogonal to the contour can be found from (2.2), and when $\rho = 1$ can be written as

$$\frac{\partial(\sigma_\theta + \sigma_\rho)}{\partial\rho} = 4\frac{p(1 - \cos(2\theta - 2\omega)) - \sigma_\theta(1 - m\cos(2\theta))}{1 - 2m\cos(2\theta) + m^2}.$$

Here σ_θ is the found from (2.3).

As in (2.6) we can write the derivative along the tangent to the contour from (2.4)

$$\frac{\partial\sigma_\theta}{\partial s} = \frac{\partial\sigma_\theta}{\partial\theta}\frac{\partial\theta}{\partial s}. \tag{2.7}$$

The derivative of σ_θ with respect to the coordinate θ can be found from (2.3), namely,

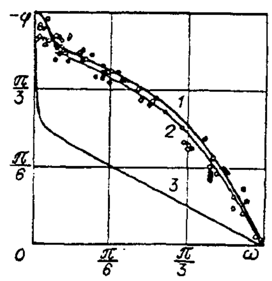

Fig. 2

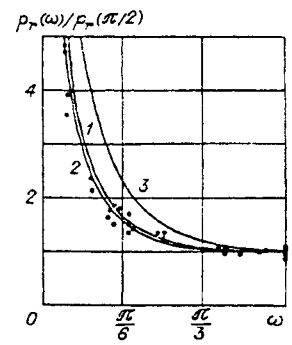

Fig. 3

$$\frac{\partial\sigma_\theta}{\partial\theta} = 4\,\frac{p\sin(2\theta - 2\omega) - \sigma_\theta m\sin(2\theta)}{1 - 2m\cos(2\theta) + m^2}. \tag{2.8}$$

It is further necessary to determine the factors in (2.6) and (2.7)

$$\frac{\partial\rho}{\partial n} = \frac{\partial\rho}{\sqrt{(\partial x)^2 + (\partial y)^2}} = \frac{1}{\sqrt{(\partial x/\partial\rho)^2 + (\partial y/\partial\rho)^2}}; \tag{2.9}$$

$$\frac{\partial\theta}{\partial s} = \frac{\partial\theta}{\sqrt{(\partial x)^2 + (\partial y)^2}} = \frac{1}{\sqrt{(\partial x/\partial\theta)^2 + (\partial y/\partial\theta)^2}}. \tag{2.10}$$

From (2.1) we have

$$x = c\left(\rho + \frac{m}{\rho}\right)\cos(\theta),\ y = c\left(\rho - \frac{m}{\rho}\right)\sin(\theta). \tag{2.11}$$

Substituting the results of differentiating expressions (2.11) into (2.9) and (2.10) with $\rho = 1$ we obtain

$$\frac{\partial\rho}{\partial n} = \frac{\partial\theta}{\partial s} = \frac{1}{\sqrt{a^2\sin^2(\theta) + b^2\cos^2(\theta)}} = \frac{1}{c\sqrt{1 - 2m\cos(2\theta) + m^2}}.$$

After determining all the necessary derivatives, using them in (2.4) and carrying out some reduction we can write the following formula for the modulus of the gradient σ_1:

$$|\mathrm{grad}\,\sigma_1| = \left(\frac{(4p(1 - \cos(2\theta - 2\omega)) - \sigma_\theta(5 - 4m\cos(2\theta) - m^2))^2}{c^2(1 - 2m\cos(2\theta) + m^2)^3} + \frac{(4p\sin(2\theta - 2\omega) - 4\sigma_\theta m\sin(2\theta))^2}{c^2(1 - 2m\cos(2\theta) + m^2)^3}\right)^{1/2}. \tag{2.12}$$

From (1.1), using (2.3) and (2.12), we have for the relative gradient

$$g_1 = \left(\frac{(4(1 - \cos(2\theta - 2\omega))\,(p/\sigma_\theta) - (5 - 4m\cos(2\theta) - m^2))^2}{c^2(1 - 2m\cos(2\theta) + m^2)^3} + \frac{(4\sin(2\theta - 2\omega)\,(p/\sigma_\theta) - 4m\sin(2\theta))^2}{c^2(1 - 2m\cos(2\theta) + m^2)^3}\right)^{1/2}. \tag{2.13}$$

When determining the relative gradient we also used another method of calculating the derivative $\partial\sigma_\theta/\partial n$, which consists of the following. Using the second Kolosov formula we find a relation for the difference between the stresses σ_θ and σ_ρ. Adding it to (2.2) we obtain an equation for σ_θ, from which, by differentiation, we obtain the required expression for $\partial\sigma_\theta/\partial n$. However, in view of its length we will not give it here. The method considered in the previous pages, which does not require the use of the second Kolosov formula and which considerably simplifies the calculation of the derivative $\partial\sigma_\theta/\partial n$ and the relative gradient for problems on stress concentration, is much more convenient.

Determination of the Point where Fracture Begins, Its Direction and the Limiting Load. Expression (2.13) for g_1 must be substituted into (1.2) and the point on the contour where the maximum effective stress σ_e is reached must be obtained. It is at this point that fracture should begin when the condition $\sigma_e = \sigma_B$ is satisfied. This procedure was carried out on a computer. We will denote the coordinate of the point at which the maximum of σ_e is reached by θ_r. The tangent of the angle γ between the abscissa axis and the ry emerging from the center of the opening to the point on the contour with coordinate $\theta = \theta_r$ (see Fig. 1), using (2.11), can be written in the form

$$\mathrm{tg}(\gamma) = \frac{1 - m}{1 + m}\,\mathrm{tg}(\theta_r).$$

Consequently,

$$\gamma = \mathrm{arctg}\left(\frac{1 - m}{1 + m}\,\mathrm{tg}(\theta_r)\right)$$

Knowing the point where fracture begins we can now determine the direction in which it develops. We will assume that the fracture occurs along the normal to the contour. In view of the orthogonality of the contour of the curvilinear coordinate ρ, the tangent of the angle φ between the abscissa axis and the normal to the contour is (see Fig. 1)

$$\mathrm{tg}(\varphi) = \frac{\partial y/\partial\rho}{\partial x/\partial\rho}.$$

After differentiating (2.11) and putting $\rho = 1$ we obtain

$$\mathrm{tg}(\varphi) = \frac{1 + m}{1 - m}\,\mathrm{tg}(\theta_r).$$

Consequently,

$$\varphi = \mathrm{arctg}\left(\frac{1 + m}{1 - m}\,\mathrm{tg}(\theta_r)\right). \tag{2.14}$$

Finally, we determine the breaking load. Knowing θ_r and using (2.3) we obtain the value of the first principal stress

$$\sigma_1 = \alpha_r\, p, \tag{2.15}$$

where α_r is understood to mean the stress intensity factor at this point

$$\alpha_r = \frac{1 - 2\cos(2\theta_r - 2\omega) - m^2 + 2m\cos(2\omega)}{1 - 2m\cos(2\theta_r) + m^2}.$$

After substituting (2.15) into (1.2), from the stability condition $\sigma_e = \sigma_B$ we have the following expression for the nominal limiting stress:

$$p_r = \frac{1}{\alpha_r}\sigma_B(1 - \beta + \sqrt{\beta^2 + L_1 g_1}). \tag{2.16}$$

Comparison of the Calculated and Experimental Data. An analysis of the literature showed that it is possible to compare the results of calculations obtained using the gradient stability condition with existing experimental data on the fracture of plane specimens with inclined cracks. Since actual cracks differ from the mathematical cuts of zero width, a crack cannot be modelled by a mathematical cut, but an elliptical opening of narrow width is physically more justified and more convenient for using the gradient condition for fracture (1.2), (1.3).

Experimental data on the fracture of plane specimens with inclined cracks made of polymethylmetacrylate are given in [13]. A value of the crack resistance $K_{1c} = 1.37$ MPa $m^{1/2}$ is given. But, unfortunately, no value of the breaking strength σ_B and the parameter β are given for this material. It was assumed that the breaking strength is equal to the critical circumferential stress p_c on the critical radius c = 0.0508 mm, given in [13]. With this assumption it is easy to determine that $\sigma_\beta = 76.6$ MPa. This value agrees with the data given in [14]. Knowing K_{1c} and σ_β and using (1.4) we can calculate the characteristic dimensions in the gradient condition for fracture $L_1 = 0.203$ mm. By processing the experimental results given in [15] we obtain a value of the parameter β close to zero for polymethylmetacrylate. Hence we will assume that $\beta = 0$.

The small circles in Fig. 2 show the values of the angle φ between the initial fracture line and the direction of propagation of the crack, determined in [13], for cuts close in form to an elliptical opening (a = 17.78 mm and b = 0.127 mm). Curve 1 in this figure represents the results of a calculation using (2.14), for the above parameters, of the angle φ between the abscissa axis and the direction in which the crack develops from a contour of an elliptical opening ($a/L_1 = 87.5$ and $b/L_1 = 0.625$). One can see that there is fairly good agreement between the calculated and experimental data. If we take into account the increase in the width of the cut before fracture, the calculated values of φ will lie between Curves 1 and 2. Curve 2 was obtained for b = 0.197 mm, which corresponds to an increase in the width of the cut before fracture with $\omega = \pi/2$ and E = 2940 MPa for polymethylmetacrylate [15].

The dark points in Fig. 2 show the values of the angle φ found in [16] for specimens of polyurethane. Unfortunately, the characteristics of this material and the width of the cuts required for calculations are not given in [16]. Note that Curve 1 also describes the experimental data for polyurethane quite well. This also applies not only to the angle φ but also to the nominal limiting stress p_r, the normalized experimental values of which are shown in Fig. 3 by the points, while Curves 1 in Figs. 2 and 3 were obtained for the same parameters. Curve 2 in Fig. 3 differs from Curve 1 in the fact that it was drawn for the results of calculations with $\beta = 0.5$ and better corresponds to the experimental data.

Hence, using the gradient condition for the fracture strength (1.2), (1.3) we can describe the experimental results on the fracture of plane specimens with inclined cracks given in the literature. Of course, these results can also be described using other criteria of classical fracture mechanics. However, the gradient condition for fracture is more universal and enables one to estimate the breaking strength of bodies with defects of different configurations, not just cracks, much more simply than when using other conditions [17]. We can thus expect new results in this area that are interesting from both the practical and scientific points of view.

It is useful to note that the use of the gradient condition for fracture at the point where σ_θ and not σ_e is a maximum on the contour of the opening for crack-like defects leads to disagreement between the calculated and experimental values (curves 3 in Figs. 2 and 3, obtained for the same parameters as curve 1).

When we attempted to use not the modulus of the gradient $|\mathrm{grad}\sigma_1|$ but its projection on the normal to the contour $|\partial\sigma_1/\partial n|$ in the gradient condition for fracture we obtained absurd results. For example, for the parameters given earlier and $\omega = \pi/2$ it turned out that the angle φ is equal to $-55°$, and not zero, as follows from consideration of symmetry. The breaking load here turned out to be one-seventh of that obtained using the well-known criteria of fracture mechanics.

The above examples and the results of calculations show that the initial formulation of the gradient condition for the breaking strength in the form (1.2), (1.3) is correct. The method considered enables us, without appreciable difficulties, to apply it not only to the problem of the stretching of plates with elliptic openings, but also for other forms of stress concentrators. Hence, using the gradient condition of the breaking strength we can obtain information on the breaking load, and the point and direction of a fracture for a wide range of problems, including those areas where the use of existing criteria of crack mechanics and the classical criteria of breaking strength are problematical.

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